Exercise 56

Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Solution

Use a logarithm to bring n down from the exponent. Assume x is a positive number.

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} \\ &= \lim_{n\to\infty} e^{n\ln\left(1+\frac{x}{n}\right)} \\ &= \exp\left[\lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)\right]\end{aligned}$$

Make the substitution $z = x/n$, or $n = x/z$. Note that as $n \to \infty$, $z \to 0$.

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \exp\left[\lim_{z\to 0}\left(\frac{x}{z}\right)\ln(1+z)\right] \\ &= \exp\left[x\lim_{z\to 0}\frac{\ln(1+z)}{z}\right]\end{aligned}$$

Use the result of the previous exercise.

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \exp[x(1)] \\ &= e^x\end{aligned}$$